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Nonparametric Estimation in Heath-Jarrow-Morton Term Structure Models Driven by Fractional Levy Processes

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ABSTRACT
We study the asymptotic theory of nonparametric estimation of the term structure's volatility for a class of one factor Heath-Jarrow-Morton term structure models driven by fractional Levy processes. This class of models is important, as it captures, as a special case, all term structure models where the short term interest rate represents a time-homogeneous univariate fractional diffusion with jumps in the equivalent risk neutral economy.

KEYWORDS
Heath-Jarrow-Morton term structure model, forward rate, fractional Levy process, semimartingale, non-Markov property, hyperbolic Levy motion, jumps, heavy tails, long memory, yield curve, stochastic volatility, kernel estimate, local time, occupation time, Ito-Tanaka formula, Monte Carlo, particle filtering.

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1. Introduction and Preliminaries

Heath, Jarrow and Morton [19] presented a general framework for modeling term structure of interest rates which nests most other models as special cases. In their framework, the dynamics of the term structure and the prices of derivative instruments depend upon the initial term structure and the forward rate volatility functions. So it is desirable to estimate the forward rate volatilities. Nonparametric estimation of forward rate volatility of standard HJM model driven by Brownian motion was studied by Jeffrey, Linton and Nguyen [22]. Bishwal [6] studied nonparametric estimation in HJM model driven by fractional Brownian motion. Bishwal [9] studied parameter estimation in stochastic volatility models with both Gaussian and non-Gaussian noises.

The insufficiency of diffusion models to explain some empirical properties of term structure has led to the development of jump- diffusion model based on Levy processes. Eberlein and Raible [14] introduced term structure model driven by Levy process by

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extending the Heath-Jarrow-Morton model which ia a particular case of the model studied by Bjork *et al.* [11]. Kuchler and Naumann [23] showed that the processes are Markovian if and only if the volatility factorises. We study nonparametric estimation of forward rate volatility driven by fractional Levy processes and study the asymptotic behavior of the estimator for high-frequency data. Our results hold for compound Poisson processes, bilateral gamma processes and in particular, variance gamma process.

It is known that the short rate in Markovian if and only if the volatility structure has either the Vasicek or the Ho-Lee form. Because of the no arbitrage restriction, in the risk neutral world, the drift function of the forward curve evolution is a function of the volatility structure. For yield curve evolution, this no arbitrage restriction also imposes that the drift function is a function of the yield volatility. Together with the knowledge of the market price of risk, the dynamics of the yield curve under the real world measure can be recovered. So for pricing based on HJM model, we only need to estimate the volatility structure.

In the original HJM models, a finite source of Brownian motions serve as the source of randomness in the economy and drive the dynamics of the whole yield curves. Empirical data shows that the yield curves have jumps.

Fractional Levy Process (FLP) is defined as

$$M_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dL_s, \quad t \in \mathbb{R}, \ 0 < H < 1,$$

where $\{L_t, t \in \mathbb{R}\}$ is a Levy process on $\mathbb{R}$ with mean zero with finite variance: $E(L_1) = 0, \ E(L_1^2) < \infty$ and the covariance of the process is given by

$$\text{cov}(M_{H,t}, M_{H,s}) = \frac{E(L_1^2)}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

Now we concentrate on the *fundamental semimartingale* behind the model. Define

$$\begin{aligned} \kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H}, \\ \eta_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \quad v_t \equiv v_t^H := \eta_H^{-1}t^{2-2H}, \quad \mathcal{M}_t^H := \int_0^t k_H(t,s)dM_s^H. \end{aligned}$$

Recall that since a Radon-Nikodym derivative process is always a martingale, in order to use Girsanov theorem for Brownian motion, a central problem for FLP is how to construct an appropriate martingale which generates the same filtration, up to sets of measure zero, called the *fundamental semimartingale*.

The natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the FLP M^H since $M_t^H := \int_0^t K_H(t,s)d\mathcal{M}_s^H$ holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \quad 0 \le s \le t.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a stochastic basis on which is defined the Ornstein-Uhlenbeck process $\{X_t\}$ satisfying the Itô stochastic differential equation

$$dX_t = \theta X_t dt + dM_{H,t}, \quad t \ge 0,$$

where $\{M_t^H\}$ is a fractional Levy process with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta < 0$ is the unknown parameter to be estimated on the basis of continuous observation of the process $\{X_t\}$ on the time interval $[0, T]$. For the standard Ornstein-Uhlenbeck process, minimum contrast estimation is studied in Bishwal [4] and Bayes estimation is studied in Bishwal [7].

Observe that $X_t = \int_{-\infty}^t e^{\theta(t-s)} dM_{H,s},\ t \geq 0$. Define $Q_t := \frac{d}{dv_t}\int_0^t k_H(t,s)X_s ds$ and $Z_t := \int_0^t k_H(t,s)dX_s$. Then $Q_t = \frac{\eta_H}{2(2-2H)}\left\{t^{2H-1}Z_t + \int_0^t s^{2H-1}dZ_s\right\}$. We have the equivalent semimartingale representation: (i) Z is the fundamental semimartingale associated with the process X, (ii) Z is a $(\mathcal{F}_t)$ -semimartingale with the decomposition $Z_t = \theta\int_0^t Q_s dv_s + \mathcal{M}_t^H$, (iii) X admits the representation $X_t = \int_0^t K_H(t,s)dZ_s$, (iv) The natural filtration $(\mathcal{Z}_t)$ of Z and $(\mathcal{X}_t)$ of X coincide.

We concentrate on our observations (data set) now. Note that for equally spaced data (homoscedastic case) $v_{t_k} - v_{t_{k-1}} = \eta_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[k^{2-2H} - (k-1)^{2-2H}],\quad k = 1, 2, \cdots, n$. For $H = 0.5$, $v_{t_k} - v_{t_{k-1}} = \eta_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[k^{2-2H} - (k-1)^{2-2H}] = \frac{T}{n},\quad k = 1, 2, \ldots, n$.

We have

$$
\begin{aligned}
Q_t &= \frac{d}{dv_t}\int_0^t k_H(t,s)X_s ds = \kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}X_s ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}X_s ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}X_s ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}X_s ds.
\end{aligned}
$$

The process Q_t depends continuously on X_t and therefore, the discrete observations of X does not allow one to obtain the discrete observations of Q. The process Q can be approximated by

$$\widetilde{Q}(n) = \kappa_H^{-1}\eta_H n^{2H-1}\sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H}X_j.$$

It is easy to show that $\widetilde{Q}_n \to Q_n$ almost surely as $n \to \infty$, see Tudor and Viens [28]. Define a new partition $0 \leq r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k,\quad k = 1, 2, \cdots, n$. Define

$$\widetilde{Q}_{t_k} = \kappa_H^{-1}\eta_H t_k^{2H-1}\sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k} - r_j)^{-1/2-H}X_{r_j}(r_j - r_{j-1}), k = 1, 2, \cdots, n.$$

It is easy to show that $\widetilde{Q}_{t_k} \to Q_t$ a.s. as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$. We use this approximation of observation in the calculation of our estimators. The nonlinear fractional SDE driven by fractional Levy process is state space transform of the fractional Ornstein-Uhlenbeck process, see Buchmann and Kluppelberg [12] for fBm driven SDE models.

Local Time for Fractional Levy Process:

Recently Ichiba *et al.* [20, 21] studied generalized fractional Brownian motion (GFBM). A generalized fractional Brownian motion is a Gaussian self-similar process whose increments are not necessarily stationary. It appears in the scaling limit of a shot-noise process with a power law shape function and non-stationary noises with a power law variance function. They studied semimartingale properties of the mixed process made up of an independent Brownian motion and a GFBM for the persistent Hurst parameter. It would be interesting to extend the current paper to GFBM noise.

Maximum quasi-likelihood estimation of unknown parameters in the unobserved volatility process in fractional Levy stochastic volatility model having fractional Levy process as the driving term which include jumps and long memory, was studied in Bishwal [5]. Maximum quasi-likelihood estimation in SPDE driven by fractional Levy processes was studied in Bishwal [8]. Bishwal [10] studied interest rate derivatives for the fractional Cox-Ingersoll-Ross model.

A fBm has a local time process $l(x,t), x \in \mathbb{R}, t \geq 0$ that is jointly continuous in x and t. The self similarity of the fBm implies the scaling property of the local time process. $\lim_{t\to\infty} l(x,t) = \infty$ for all $x \in \mathbb{R}$. The process $l(x,t)$ has moments of all orders finite and uniformly bounded in all real x and all t in a compact set.

Local Time for Levy Process:

For symmetric α stable Levy process X with $\alpha \in (1,2]$, Salminen and Yor showed which powers of local times are semimartingales:

$$|X_t - x|^{\alpha-1} = |x|^{\alpha-1} + N_t^x + c_1 L_t^x.$$

where N is a martingale with

$$\langle N\rangle_t = c_2 \int_0^t \frac{ds}{|X_s - x|^{2-\alpha}}.$$

For $\alpha < \gamma < \alpha$, $|X_t - x|^\gamma$ is a submartingale. For $0 < \gamma < \alpha - 1$, $|X_t - x|^\gamma$ is not a semimartingale but for $(\alpha-1)/2 < \gamma < \alpha - 1$, $|X_t - x|^\gamma$ is a Dirichlet process with the canonical decomposition

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}.$$

where $N_t^{(\gamma)}$ is a martingale and $A_t^{(\gamma)}$ has zero quadratic variation.

Itô formula for semimartingales:

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{\sigma^2}{2} f''(X)ds + \sum_{0<s\leq t} (f(X_s) - f(X_{s-}) - f''(X_{s-})\Delta X_s).$$

Occupation time formula:

$$\int_0^t f(X_s)ds = \int_{-\infty}^{\infty} f(x)L_t^x dx.$$

for every Borel function f on $\mathbb{R}$.

Occupation measure: Thus L_t^x is Radon-Nikodym derivative of the occupation measure of X with respect to Lebesgue measure on $\mathbb{R}$.

$$\int_0^t f(X_s)d\langle X^c\rangle = \int_{-\infty}^{\infty} f(x)\lambda_t^x dx.$$

$d\langle X^c\rangle = ds$, i.e., X^c is a Brownian motion, then L and λ coincide.

Tanaka Formula:

$$v(X_t - x) = v(x) + N_t^x + L_t^x.$$

For standard Brownian motion, $v(x) = |x|$.

Estimation of the Diffusion Coefficient

Suppose the process X satisfies

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t$$

where a is a bounded function, twice continuously differentiable, with bounded derivatives, σ is an unknown function with three continuous and bounded derivatives such that there exists two constants k and K with $0 < k \leq \sigma(x) \leq K$. Define

$$s(x) = \int_0^x \sigma^{-1}(u)du$$

and $g = s^{-1}$. The process $Y_t = s(X_t)$ satisfies

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, X_0 = x_0$$

where $b = (\frac{a}{\sigma}) \circ g - \frac{1}{2}\sigma' \circ g$ which is a bounded function of class C^2.

Local time of Y in x during $[0, t]$ is defined as

$$L_t(x) = \lim_{\delta\to 0}\frac{1}{\delta}\int_0^t I_{\{|Y_s - x|<\delta\}}ds.$$

and its discrete approximation is defined as

$$L_{n,t}(x) = \frac{1}{2nh}\sum_{i=1}^n I_{\{|Y_{t_{i-1}} - x|<\delta\}}, \quad L_{n,T}(x) = \sum_{i=1}^n I_{x,h}(y) = \sum_{i=1}^n I_{\left(\frac{|y-x|}{h}<1\right)}.$$

Let h be the spatial discretization step, the bandwidth. Florens-Zmirou [16] used discrete approximation of the local time to estimate the diffusion coefficient:

$$\widehat{\sigma}^2_{n,T}(x) = \frac{\sum_{i=1}^n I_{x,h}(Y_{t_{i-1}})(\Delta Y_{t_i})^2}{\sum_{i=1}^n I_{x,h}(Y_{t_{i-1}})\Delta t_i}.$$

The estimator has the following properties:

Theorem 1.1
1) As $nh^4 \to 0$ *as* $n \to \infty, L_{n,T}(x) \to^{L_2} L(x)$.
2) As $(nh^2)^{-1}\log n \to 0$ *as* $n \to \infty, L_{n,T}(x) \to^{a.s.} L(x)$.
3) Thus $\widehat{\sigma}^2_{n,T}(x)$ *is a consistent estimator of* $\sigma^2(x)$.
4)

$$\sqrt{nh}\left(\frac{\widehat{\sigma}^2_{n,T}(x)}{\sigma^2(x)} - 1\right) \to^d L(x)^{-1/2} Z$$

where Z *is standard normal independent of the local time* $L(x)$.
5) With random norming as $nh^4 \to 0$ *as* $n \to \infty$,

$$\sqrt{N_x^n}\left(\frac{\widehat{\sigma}^2_{n,T}(x)}{\sigma^2(x)} - 1\right) \to^d \sqrt{2}Z \quad \text{where} \quad N_x^n = nL_x^n.$$

Remarks: 1) $nh^4 \to 0$ condition can be relaxed by using Itô formula.
2) $(nh^2)^{-1}\log n \to 0$ condition can be relaxed by using Itô formula.

Monte Carlo Estimate of Local Time:

$$L_{m,n,t}(x) = \frac{1}{2mnh}\sum_{i=1}^n\sum_{j=1}^m I_{\{|Y_{t_{i-1},j}-x|<\delta\}}.$$

Theorem 1.2 $L_{m,n,T}(x) \to^P L(x)$ as $nh^4 \to 0$ as $n \to \infty$ and $m \to \infty$.

Proof: First observe that by the LLN, $L_{m,n,t}(x) \to^P L_{n,t}(x)$ as $m \to \infty$.
The expansion of the transition density p_t in y is given by

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\frac{1}{\sigma(y)}\exp\left(-\frac{(s(y)-s(x))^2}{2t}\right)U_t(s(x),s(y))$$

with

$$U_t(x,y) := H_t(x,y)\exp[A(y)-A(x)],$$

$$H_t(x,y) := E\left[\exp\left(-t\int_0^1 f(x+z(x-y)+\sqrt{t}B_z)dz\right)\right]$$

with B is a Brownian bridge, A an integral of b and $f = \frac{1}{2}(b^2 + b')$.

Following Friedman (1975), (see Nakamura and Zheng [24], and Gobet [18])

$$p_t(x,y) \le \frac{C}{\sqrt{t}} \exp\left(-\frac{(y-x)^2}{2t}\right), \quad \frac{\partial p_t(x,y)}{\partial x} \le \frac{C|x-y|}{t^{3/2}} \exp\left(-\frac{(y-x)^2}{2t}\right).$$

The MC estimate of this expansion of the transition density is given by

$$p_{m,t}(x,y) := \frac{1}{\sqrt{2\pi t}} \frac{1}{\sigma(y)} \exp\left(-\frac{(s(y)-s(x))^2}{2t}\right) U_{m,t}(s(x), s(y))$$

with

$$U_{m,t}(x,y) := H_{m,t}(x,y) \exp[A(y) - A(x)],$$

$$H_{m,t}(x,y) := \frac{1}{m} \sum_{j=1}^{m} \left[\exp\left(-t \int_0^1 f_j(x + z(x-y) + \sqrt{t} B_z) dz\right)\right]$$

where B is a Brownian bridge, A an integral of b and $f_j = \frac{1}{2}(b_j^2 + b_j')$, $b_j = b(X_{t,j})$ and $X_{t,j}$ is the j-th simulated path of $X_t, j = 1, 2, \ldots, m$ by, for example, exact simulation method, see Beskos and Roberts [3].

For simplicity take $x = 0$ and $t = 1$. We have

$$L_{m,n,t}(x) := \frac{1}{m} \sum_{i=1}^{n} \sum_{j=1}^{m} I_{\{|Y_{t_{i-1},j} - x| < \delta\}}.$$

Let

$$\xi_{i,m} := \frac{1}{m} \frac{1}{2nh} \sum_{j=1}^{m} I_{\{|Y_{t_{i-1},j}| < h\}}, \quad \eta_{i,m} := L_{(i+1)/n,m} - L_{i/n,m}.$$

We have

$$\sum_{i \ne k} E^{x_0}(\xi_{i,m}\xi_{k,m}) = 2 \sum_{i<k} \frac{1}{4n^2h^2} \sum_{j=1}^{m} \int_{-h}^{h} \int_{-h}^{h} p_{i/n,m}(x_0, x) p_{(k-i)/n,m}(x,y) dx dy.$$

Let

$$R^1_{m,n} := \left| \sum_{i \ne k} E^{x_0}(\xi_{i,m}\xi_{k,m}) - \frac{2}{n^2} \sum_{i<k} \sum_{j=1}^{m} p_{i/n,m}(x_0, 0) p_{(k-i)/n,m}(0,0) \right|.$$

It can be shown that

$$R^1_{m,n} = O(\sqrt{n}h^2) + O(h), \quad E^{x_0}(\xi_{i,m})^2 \le \frac{2h}{4n^2h^2} p^*_{i/n,m}(x_0, h),$$

$$\sum_{i=1}^{n} E^{x_0}|\xi_{i,m}\eta_{i,m}| = O(\frac{1}{n^{3/4}h^{1/2}}).$$

Let

$$R^2_{m,n} := \left| \sum_{i<j} E^{x_0}(\xi_{i,m}\eta_{j,m}) - \frac{1}{n}\sum_{i<j}\sum_{j=1}^{m}\int_{j/n}^{(j+1)/n} p_{i/n,m}(x_0,0)p_{s-(k-i)/n,m}(0,0)ds \right|.$$

It can be shown that

$$R^2_{m,n} = O(\sqrt{n}h^2) + O(h), \quad \sum_{i=1}^{n} E^{x_0}(\eta_{i,m})^2 = O(\frac{1}{nh}).$$

Let

$$I_n = E^{x_0}[\sum_i I_{|Y_{t_{i-1},j}|<h}[n(Y_{(i+1)/n} - Y_{i/n})^2 - \sigma^2(0)]] =: \sum_{i,k} I_{m,n}(i,k)$$

where

$$I_{m,n}(i,k) = \int_{-h}^{h}\int_{-h}^{h} p_{i/n,m}(x_0,x)dx \left[\int_{-\infty}^{\infty} p_{1/n,m}(x,y)[n(y-x)^2 - \sigma(0)^2]dy\right.$$

$$\left. \times \int_{-h}^{h}\int_{-h}^{h} p_{(k-i-1)/n,m}(y,z)dz \int_{-\infty}^{\infty} p_{1/n,m}(z,v)[n(v-z)^2 - \sigma(0)^2]dv\right]$$

Let

$$I_{m,n}(\alpha,i,k) = \int_{-h}^{h}\int_{-h}^{h} p_{i/n,m}(x_0,x)dx \left[\int_{-n^\alpha}^{n^\alpha} p_{1/n,m}(x,y)[n(y-x)^2 - \sigma(0)^2]dy\right.$$

$$\left. \times \int_{-h}^{h}\int_{-h}^{h} p_{(k-i-1)/n,m}(y,z)dz \int_{-n^\alpha}^{n^\alpha} p_{1/n,m}(z,v)[n(v-z)^2 - \sigma(0)^2]dv\right].$$

We have

$$|I_{m,n}(i,k) - I_{m,n}(\alpha,i,k)| = O(\exp(-Cn^{2\alpha})), \quad \exp[A(x+u/\sqrt{n}) - A(x)] \le C\exp C|u|,$$

$$\exp H_{1/n}(x+u/\sqrt{n}) = O(1), \quad |n(g(x+u/\sqrt{n}) - g(x))^2 - g'^2(0)| \le C(u^2+1).$$

Consider the martingale

$$M_t^n = \sum_{i=0}^{[nt]-1} \sqrt{\frac{n}{2h}} I_{|Y_{t_{i-1},j}-x|<h}[n(Y_{(i+1)/n} - Y_{i/n})^2 - \sigma^2(x)/n].$$

It can be shown that $M_t^n \to M_t$ with increasing process $\langle M\rangle_t = 2\sigma^4(x)L_t(x)$. Then we can write $M_t = B_{2\sigma^4(x)L_t(x)}$ where B_t is a Brownian motion. If $\tau_t := \{u : 2\sigma^4(x)L_t(x) > t\}$ then $B_t = M_{\tau_t}$. By Knight's theorem B_t and W_t are independent Brownian motions. Note that local time at a stopping time (Knight's Theorem) has a connection to Bessel process.

Let the martingale difference sequence be defined as

$$m_{i+1} := \sqrt{\frac{n}{2h}} I_{|Y_{t_{i-1},j}-x|<h}[n(Y_{(i+1)/n} - Y_{i/n})^2 - \sigma^2(x)/n]$$

$$w_{i+1} := W_{(i+1)/n} - W_{i/n}, \quad M_t^n = \sum_{i=0}^{[nt]-1} m_{i+1}, \quad W_t^n = \sum_{i=0}^{[nt]-1} w_{i+1}.$$

Using Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, Itô's formula as $nh^3 \to 0$ as $n \to \infty$,

$$\sum_{i=0}^{[nt]-1} E^{i,n} m_{i+1} \to^P 0, \; \sum_{i=0}^{[nt]-1} E^{i,n} m_{i+1}^2 \to^P 2\sigma^4(x)L_t(x),$$

$$\sum_{i=0}^{[nt]-1} E^{i,n} m_{i+1}^3 \to^P 0, \; \sum_{i=0}^{[nt]-1} E^{i,n} w_{i+1} m_{i+1} \to^P 0.$$

The process $(Y_{(i+1)/n,s} - Y_{i/n,s})_{0\le s\le 1/n}$ is a semimartingale for which Burkholder-Davis-Gundy inequality can be applied. By Itô's formula

$$\begin{aligned} M_t^n &= \sum_{i=0}^{[nt]-1} \sqrt{\frac{n}{2h}} I_{\{|Y_{t_{i-1},j}-x|<h\}}[n(Y_{(i+1)/n} - Y_{i/n})^2 - \sigma^2(x)/n] \\ &= \sum_{i=0}^{[nt]-1} \sqrt{\frac{n}{2h}} I_{\{|Y_{t_{i-1},j}-x|<h\}} \left[\int_{i/n}^{(i+1)/n} 2(Y_{(i+1)/n} - Y_{i/n}) b(Y_u) du \right. \\ &\quad \left. + \int_{i/n}^{(i+1)/n} (\sigma^2(Y_u) - \sigma^2(x)) du \right]. \end{aligned}$$

Thus

$$|E(M_t^n)| = O(1)[h^{1/2} + h^{3/2}n^{1/2}]L_{n,t}(x).$$

This converges to zero in probability as $nh^3 \to 0$ since $L_{n,t}(x) \to L_t(x)$. □

2. Term Structure Dynamics and Model Specification

Modeling the dynamics of interest rates with jumps is of recent interest since they provide a better characterization of randomness in the financial markets than diffusion models. Levy driven HJM framework introduced in Eberlein and Raible [14] represents the term-structure in terms of forward rates and for a single source of uncertainty in the bond market, introduced by the Levy process L_t, the uncertain evolution of each forward rate with fixed maturity date T satisfies the SDE

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dM_t^H$$

where

$$\alpha(t,T) = r(t) - \theta(\sigma(t,T))$$

and $r(t)$ is the short rate, $\theta(u)$ is the logarithmic moment generating function of M_1^H. Here $\{M_t^H\}$ is a fractional Levy process with Levy measure F satisfying

$$\int_{|x|>1} \exp(ux)F(dx) < \infty$$

for all u from an open interval $I = (u_0, u_1)$ including zero.

Let $P(t,T)$ be the price of a one dollar face value, default free, zero coupon bond at time t that will mature at time T. The instantaneous forward rate at time t for date T denoted by $f(t,T)$ is defined by

$$f(t,T) := -\frac{\partial \ln P(t,T)}{\partial T}$$

and the short rate is defined as

$$r(t) := f(t,t).$$

We will assume that $-\frac{\partial \ln P(t,T)}{\partial T}$ exists for all T.

The bond price satisfies the SDE

$$dP(t,T) = P(t,T)[r(t)dt + \sigma(t,T)dM_t^H]$$

which gives

$$P(t,T) = P(0,T)\beta(t)\frac{\exp\int_0^t \sigma(s,T)dM_s^H}{E\left[\exp\int_0^t \sigma(s,T)dM_s^H\right]}$$

where

$$\beta(t) = \int_0^t r(s)ds$$

is the value of the numeraire at time t and $P(T,T) = 1$.

The forward rate satisfies the equation:

$$f(t,T) = f(0,T) + \int_0^t \theta'(\sigma(s,T))\sigma_2(s,T)ds - \int_0^t \sigma_2(s,T)dM_s^H, 0 \le t \le T$$

where $\theta(v) = \log E[\exp(M_1^H)]$.

$$\theta(z) = bz + \frac{c}{2}z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx)F(dx).$$

$$f(0,T) = -\partial/(\partial T)\log P(0,T), \quad \theta'(u) := \frac{d}{du}\theta(u), \quad \sigma_2(t,T) := \partial/(\partial T)\sigma(t,T).$$

$\sigma_2(t,T)$ is called the volatility function.

The short rate $r(t) = f(t,t)$ is the instantaneously maturing forward rate at time t:

$$r(t) = f(0,t) + \int_0^t \theta'(\sigma(s,t))\sigma_2(s,t)ds - \int_0^t \sigma_2(s,t)dM_s^H.$$

The short rate process is Markovian if and only if the volatility function factorizes and $H = 0.5$:

$$\sigma_2(t,T) = \tau(t)\zeta(T)$$

for some functions $\tau(\cdot)$ and $\zeta(\cdot)$, (see Eberlein and Raible [14] and Kuchler and Naumann [23]).

We assume that

$$|E\exp(iuM_1^H)| \le C\exp(-\gamma|u|^\eta), \quad u \in \mathbb{R}$$

for some positive constants C, γ, η.

This condition is satisfied for example for Wiener processes, Normal Inverse Gaussian (NIG) Levy processes, Stable processes and Hyperbolic Levy processes. On the other hand, it does not hold for compound Poisson process, gamma processes and finite sums of independent examples of them.

Let $\sigma : [0,T] \to \mathbb{R}$ be a deterministic, continuously differentiable function. Then for almost all $\omega \in \Omega$,

$$\int_0^t \sigma(s)dM_s^H = \sigma(t)M_t^H - \int_0^t \sigma'(s)ds \;\forall t \in [0,T].$$

If L is the standard Wiener process, we have $\theta'(u) = u$ and the model satisfies the classical HJM condition on the drift coefficient of the forward rate process.

If f is a continuously differentiable function on $[0,t]$ having values in the interval (u_0, u_1) only, then

$$E\exp\left[\int_0^t f(s)dM_s^H\right] = \exp\left[\int_0^t \theta(f(s))ds\right].$$

Finally,

$$P(t,T) = E\left(\exp\left(-\int_t^T r_s ds\right) \mid \mathcal{A}_t\right).$$

When the short rate is a Markov process the bond price can be evaluated explicitly.

The discounted processes $\widetilde{P}(t,T) = \beta(t)^{-1}P(t,T)$ are martingales. The short rate satisfies the SDE

$$dr(t) = [f_2(0,t)+\theta'(\sigma(0,t))\sigma_2(0,t)]dt+\frac{\zeta'(t)}{\zeta(t)}[\theta(\sigma(0,t))+f(0,t)-r(t)]dt-\sigma_2(t,t)dM_t^H.$$

Our aim in this paper is to estimate the term structure's volatility which is given by the forward rate volatility $\sigma(t,T)$. We observe the whole term structure in discrete points in time. Typically estimation of the forward rate curve is more difficult than yield curve. The yield at time t with maturity date T, denoted by $y(t,T)$ is fined by

$$y(t,T) = -\frac{1}{T-t}\ln P(t,T).$$

Thus forward rate and yield curve are related by

$$f(t,T) = y(t,T) + (T-t)\frac{\partial}{\partial T}y(t,T).$$

We will estimate the term structure yield volatility instead of the forward volatility.

Yield curve satisfies the SDE

$$dy(t,T) = m(t,T) + \eta(t,T)dM_t^H$$

where

$$\eta(t,T) = \frac{1}{T-t}\int_0^T \sigma(t,s)ds, \quad m(t,T) = \frac{y(t,T)-r(t)}{T-t} + r(t)\eta(t,T) + \frac{1}{2}\eta^2(t,T).$$

Following are the fractional Levy versions of most popular short rate models.

1. Fractional Levy Vasicek Model:

$$dV_t = (b+\beta r_t)dt + \sigma\sqrt{r_t}dM_t^H$$

2. Fractional Levy Cox-Ingersoll-Ross (CIR) Model:

$$dV_t = (b+\beta r_t)dt + \sigma\sqrt{r_t}dM_t^H$$

3. Fractional Levy Dothan Model:

$$dV_t = (b+\beta r_t)dt + \sigma r_t dM_t^H$$

4. Fractional Levy Black-Derman-Toy Model:

$$dV_t = \beta(t)r_t dt + \sigma(t)r_t dM_t^H$$

5. Fractional Levy Black-Karasinksi Model: $l_t = \log r_t$.

$$dV_t = (b(t)+\beta(t)l_t)dt + \sigma(t)dM_t^H$$

6. Fractional Levy Ho-Lee Model:

$$dV_t = b(t)dt + \sigma dM_t^H$$

7. Fractional Levy Hull-White (Extended Fractional Levy Vasicek) Model:

$$dV_t = (b(t) + \beta(t)r_t)dt + \sigma_t dM_t^H$$

8. Fractional Levy Hull-White (Extended CIR) Model:

$$dV_t = (b + \beta r_t)dt + \sigma_t \sqrt{r_t} dM_t^H.$$

3. Yield Curve Volatility Structure Estimation

Consider the Markov diffusion process

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad t \in [0, T].$$

Bandi and Phillips [2] proposed the following two kernel estimators of μ and σ:

$$\widehat{\mu}_{n,T}(x) = \frac{\sum_{i=1}^n K_{x,h}(Y_{t_{i-1}})\Delta Y_{t_i}}{\sum_{i=1}^n K_{x,h}(Y_{t_{i-1}})\Delta t_i}, \quad \widehat{\sigma}^2_{n,T}(x) = \frac{\sum_{i=1}^n K_{x,h}(Y_{t_{i-1}})(\Delta Y_{t_i})^2}{\sum_{i=1}^n K_{x,h}(Y_{t_{i-1}})\Delta t_i}$$

where the kernel $K_{x,h}(y) := K\left(\frac{y-x}{h}\right)$. See Bandi and Nguyen [1] for extension to jump diffusion models.

We use the kernel method to estimate the yield curve volatility $\eta(t, T)$ in the non-Markov non-semimartingale model. The kernel estimator is shown to be consistent and mixed normally distributed. We also estimate the rate of convergence. Our observations are the short term interest rates $r(t_i), i = 1, 2, \cdots, n$ and the yield transitions $\Delta\widetilde{y}(t_i, \tau) := \widetilde{y}(t_{i+1}, \tau) - \widetilde{y}(t_i, \tau), i = 1, 2, \cdots, n$. For simplicity it is assumed that all time intervals are equally spaced, that is, $\Delta t := t_{i+1} - t_i$ for all i. Denote $\widetilde{\eta}(t, \tau) = \eta(r(t), \tau)$.

$$\widetilde{m}(t, \tau) = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} E[(\widetilde{y}(t + \Delta t, \tau)) - \widetilde{y}(t, \tau))|\mathcal{F}_t] \right).$$

It is apparent that

$$\widetilde{\eta}(t, \tau)^2 = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} E[(\widetilde{y}(t + \Delta t, \tau)) - \widetilde{y}(t, \tau))^2|\mathcal{F}_t] \right).$$

The yield volatility structure is the standard deviation of the yield curve transitions. Since we restrict volatility structures of the form $\widetilde{\eta}(t, \tau) = \eta(r(t), \tau)$, it is enough to determine the volatility structure at time t. Consequently the following approximation for the yield volatility holds:

$$\widetilde{\eta}(t, \tau)^2 \simeq \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} E[(\widetilde{y}(t + \Delta t, \tau)) - \widetilde{y}(t, \tau))^2|r_t] \right).$$

Replacing the expectation on the right hand side by the Nadaraya-Watson kernel smoothing estimator, the yield volatility estimator is given by

$$\widehat{\eta}(t,\tau)^2 = \frac{1}{\Delta t}\frac{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta\widetilde{y}(t_i,\tau))^2}{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})}$$

where $K(\cdot)$ is a symmetric kernel function that integrates to one and h is a bandwidth parameter that decreases to zero as sample size $n\to\infty$ that controls the amount of smoothing.

Replacing the expectation on the right hand side by the Nadaraya-Watson kernel smoothing estimator, the yield mean estimator is given by

$$\widehat{m}(t,\tau) = \frac{1}{\Delta t}\frac{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta\widetilde{y}(t_i,\tau))}{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})}.$$

We will us the following result to prove consistency.

Lemma 3.1 (Newey and Powell [25]) *Suppose that i) there exists some deterministic function $Q(\gamma)$ with a unique minimum at γ_0 lying in the metric space Γ; ii) $Q_{nJ}(\gamma)$ and $Q(\gamma)$ are continuous; iii) Γ is compact and iv)* $\sup_{\gamma\in\Gamma}|Q_{nJ}(\gamma)-Q(\gamma)| = o_P(1)$. *Define $\widehat{\gamma} = \arg\min_{\gamma\in\Gamma} Q_{nJ}(\gamma)$. Then $\widehat{\gamma}\to^P \gamma_0$.*

Theorem 3.1 *The yield volatility structure estimator is strongly consistent, i.e.,*

$$\widehat{\eta}(t,\tau)^2 - \eta(t,\tau)^2 \to 0 \ \textit{a.s.} \text{ as } \ n\to\infty, \Delta t\to 0, h_{n,T}\to 0.$$

Proof. We verify the conditions of Lemma 3.1. Observe that

$$\widehat{\eta}(t,\tau)^2 - \eta(t,\tau)^2 = \frac{1}{\Delta t}\frac{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta\widetilde{y}(t_i,\tau))^2}{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})}$$

$$-\lim_{\Delta t\to 0}\left(\frac{1}{\Delta t}E[(\widetilde{y}(t+\Delta t,\tau)) - \widetilde{y}(t,\tau))^2|r_t]\right).$$

Using Taylor expansion, we have

$$\widehat{\eta}(t,\tau)^2 - \eta(t,\tau)^2 = \frac{\int \widehat{H}_1(r,\tau,s)\widehat{\eta}(s,\tau)^2 ds}{\int \widehat{H}_2(r,\tau,s)\widehat{\eta}(s,\tau)^2 ds} - \frac{\int \widehat{H}_1(r,\tau,s)\eta(s,\tau)^2 ds}{\int \widehat{H}_2(r,\tau,s)\eta(s,\tau)^2 ds}$$

where

$$\widehat{H}_1(r,\tau,s) = \sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta\widetilde{y}(t_i,\tau))^2, \quad \widehat{H}_2(r,\tau,s) = \sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})\Delta t.$$

From Florens-Zmirou [16], we have

$$\Delta_n \sum_{i=1}^{n-1} \frac{1}{h} K(\frac{r - r_{t_i}}{h}) \to^P \bar{l}_r(t_n, r) \quad \text{as} \quad n\Delta_n^4 \to 0.$$

Assumptions
(A1) The drift and the diffusion are locally bounded

$$|\alpha(t_i) - \alpha(t_j)| + |\eta(r(t_i)) - \eta(r(t_j))| \le C|Z(t_i) - Z(t_j)|$$

and there exists some $0 < \nu < \frac{1}{2}$ such that

$$\left| \frac{\Delta_n^\nu}{\bar{L}(t,a)} \int_0^{t_n} \frac{1}{h} K(\frac{r - r_s}{h}) \alpha(s) ds \right| = O(1)$$

where Z_t is the Dirichlet process

$$dZ_t = \alpha(t)dt + \eta(r(t))dM_t^H = (\zeta(r(t)) + \phi(t))dt + \eta(r(t))dM_t^H$$

and r_t is the Dirichlet process of the form

$$dr_t = \mu(t)dt + \sigma(r(t))dM_t^H.$$

(A2) $\bar{l}(t, a) \to \infty$ as $t \to \infty$.

Theorem 3.2 *The asymptotic distribution of the yield volatility estimator is mixed normal, i.e,*

$$\sqrt{\frac{h\bar{l}(t_n, r)}{\Delta t}} \, (\widehat{\eta}(t,\tau)^2 - \eta(t,\tau)^2) \to^{\mathcal{D}} \mathcal{MN}\left(0, 4\left(\int_{-\infty}^{\infty} K(u)^2 du\right) \eta(r,\tau)^4\right)$$

as $n \to \infty$ where $\bar{l}(t_n, r)$ is the local time at the point r of the process $r(t)$ which is a measure of the absolute amount of time that the short term interest rate process spends in the vicinity of the point r over the time interval $[0, t_n]$. An estimate of the asymptotic variance of $\widehat{\eta}(t,\tau)^2$ can be obtained by replacing $\eta(\tau,\tau)^4$ with $\widehat{\eta}(\tau,\tau)^4$ and $\bar{l}(t_n, r)$ can be estimated by $\Delta t \sum_{i=1}^n K_h(r - r(t_i))$.

Proof. Consider the Dirichlet process

$$\begin{aligned} dZ_t &= \alpha(t)dt + \eta((r(t))dM_t^H \\ &= [\zeta(r(t)) + \phi(t)]dt + \eta(r(t))dM_t^H \end{aligned}$$

where $r(t)$ is also a Dirichlet process of the form

$$dr(t) = \mu(t)dt + \sigma(r(t))dM_t^H.$$

Note that

$$\widehat{\eta}(r)^2 = \frac{1}{\Delta t}\frac{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta Z(t_i))^2}{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})},\ \widehat{\zeta}(r)^2 = \frac{1}{\Delta t}\frac{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})(\Delta Z(t_i)-\phi(t_i)\Delta t)}{\sum_{i=1}^{n-1}\frac{1}{h}K(\frac{r-r_{t_i}}{h})}.$$

The local time of the Dirichlet process r at a point a over the time interval $[0,t]$ the amount of time spent by the process near a and is defined as

$$l(t,a) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t I_{\{|r_s-a|<\epsilon\}}\sigma^2(r_s)ds.$$

The discrete approximation of $l(t,a)$ is

$$l_n(t,a) = \frac{1}{2n\Delta_n}\sum_{i=1}^{n-1} I_{\{|r_{t_i}-a|<\Delta_n\}}\sigma^2(r_{t_i})(t_{i+1}-t_i).$$

Occupation Time Formula: For every Borel f, we have

$$\int_0^t f(r_s)d\langle r,r\rangle = \int_{-\infty}^{\infty} f(a)l(t,a)da.$$

The kernel K has compact support and is symmetric about 0 and is continuously differentiable.

Remark As $n\Delta_n^3 \to 0$, using random norming (discrete local time), one can obtain asymptotic normality as in Florens-Zmirou [16].

4. Stochastic Volatility Model

Let the Musiela parametrization of forward rates be given by $g(t,x) = f(t,t+x)$. Here x denotes the time to maturity as opposed to T which is time of maturity. The stochastic volatility model is the pair $(g(t,x), v_t)$ satisfying the infinite dimensional SDE

$$dg(t,x) = \left\{\frac{\partial}{\partial x}g(t,x) + \sigma(g(t,x)), v_t, x)\int_0^x \sigma(g(t,s)), v_s, s)ds\right\}dt + \sigma(g(t,x)), v_t, x)dL_t,$$
$$dv_t = a(v_t)dt + b(v_t)dM_t^H,\ t \geq 0$$

where $(M_t^H, t \geq 0)$ is a fractional Levy process.

The forward rate volatility is assumed to be an arbitrary functional of the entire forward curve as well as the variable v_t. Carmona and Tehranchi [13] studied infinite dimensional approach to HJM model. We suggest to use the branching particle algorithm to solve this nonlinear volatility estimation problem.

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